On-shell recursion relations for all Born QCD amplitudes

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ABSTRACT: We consider on-shell recursion relations for all Born QCD amplitudes. This includes amplitudes with several pairs of quarks and massive quarks. We give a detailed description on how to shift the external particles in spinor space and clarify the allowed helicities of the shifted legs. We proof that the corresponding meromorphic functions vanish at $z \to \infty$. As an application we obtain compact expressions for helicity amplitudes including a pair of massive quarks, one negative helicity gluon and an arbitrary number of positive helicity gluons.

Contents

1. Introduction

In the past years, various new methods for efficient calculations in QCD have been introduced, motivated by the relation of QCD amplitudes to twistor string theory found in [1]. In particular these methods include the diagrammatic rules of Cachazo, Svrček and Witten (CSW) [2], where tree level QCD amplitudes are constructed from vertices that are off-shell continuations of maximal helicity violating (MHV) amplitudes [3], and the recursion relations of Britto, Cachazo, Feng and Witten (BCFW) [4, 5] that construct scattering amplitudes from on-shell amplitudes with external momenta shifted into the complex plane. The BCFW recursion relations have found numerous applications in tree

level [6 – 19] and one-loop [20 – 27] calculations in QCD. Extensions to QED [28] and gravity [29 – 33] also have been considered. The relation of the BCFW method to the usual Feynman diagrams has been clarified in [34, 35] and it has been used to give a proof for the CSW construction [36]. The main advantages of the BCFW and CSW constructions are in the simplification of analytical calculations as compared to more traditional off-shell recursive methods [37].

Most of the literature related to these new methods restricts itself to the all-gluon amplitude. For calculations of multi-parton scattering amplitudes relevant for phenomenological applications at upcoming colliders such as the Large Hadron Collider (LHC) it is desirable to extent these methods towards the full particle content of the Standard Model. In the CSW approach it has been possible to include single external massive gauge bosons or Higgs bosons [38 – 41], while the BCFW recursion relations have been successfully applied to derive multigluon amplitudes involving a pair of massive scalars [9, 11]. As shown in [42] some helicity amplitudes for massive quarks can be obtained from these scalar amplitudes by Ward-identities in super-symmetric-QCD. A compact expression for amplitudes with a pair of massive scalars or quarks and an arbitrary number of positive helicity gluons has been found in [43, 13] by a combination of off-shell recursive methods and the BCFW relations.

On-shell recursion relations for amplitudes involving massive quarks have been considered in [10], but the proof is restricted to the case where the shifted particles are massless. It should be noted that expressions for shifts of massive momenta have been stated in [9]. However, Ozeren and Stirling [14] report that they were unable to construct all helicity combinations of the $\bar{t}t \to ggg$ amplitude from on-shell recursion relations. In addition, already the question of allowed helicities for the shifts of massless quarks does not appear to be completely settled in the literature [6, 7, 10, 12, 18, 19].

The purpose of this paper is to clarify the situation for on-shell recursion relations for Born QCD amplitudes. The particle content of QCD are gluons and quarks, where the latter may be massive or massless. We derive expressions for shifts of spinors for all particles — massless or not — and investigate the allowed helicity combinations of the shifted particles. Our findings can be summarised as follows: As in the massless case we have for each pair of marked particles two possibilities to shift the spinors — a holomorphic one and an anti-holomorphic one. The two marked particles must not be quarks belonging to the same fermion line. For each of the four possible helicity assignments of the two marked particles at least one shift leads to a recursion relation. The only exceptions to this rule are amplitudes involving solely massive quarks. In this case two-particle shifts are not sufficient. However, amplitudes consisting only of massive quarks and sufficient many external legs may be computed recursively from three-particle shifts.

We show that shifts of massive particles can lead to simpler recursion relations than those considered previously in the literature and use them to derive amplitudes with a pair of massive quarks, one negative helicity gluon and an arbitrary number of positive helicity gluons. Using super-symmetric Ward identities we also obtain a more compact form for the corresponding amplitudes with a pair of massive scalars than known previously.

This paper is organised as follows: In section 2 we introduce our notation together with

a short review of the colour decomposition of QCD amplitudes and an introduction to the spinor helicity formalism. Section 3 explains in detail the recursion relation for Born QCD amplitudes. This is the main result of this paper. The proof of the recursion relation is given in section 4. In section 5 we discuss applications of the recursion relation and provide examples. Section 6 contains our conclusions. In appendix A we collected information on the construction of massless spinors out of light-like four-vectors. Appendix B contains the discussion of a few exceptional cases, which are needed for the proof of the recursion relation in section 4.

2. Notation and conventions

In this section we briefly review the colour decomposition of QCD amplitudes and the spinor helicity formalism.

2.1 Colour decomposition

Amplitudes in QCD may be decomposed into group-theoretical factors (carrying the colour structures) multiplied by kinematic functions called partial amplitudes [44–48]. These partial amplitudes do not contain any colour information and are gauge-invariant objects. Although no arguments in this paper rely on colour decomposition, the examples we present are based on partial amplitudes. By convention we consider all particles as out-going.

In the pure gluonic case tree level amplitudes with n external gluons may be written in the form

$$\mathcal{A}_n(1, 2, \ldots, n) = \left(\frac{g}{\sqrt{2}}\right)^{n-2} \sum_{\sigma \in S_n/Z_n} \delta_{i_{\sigma_1} j_{\sigma_2}} \delta_{i_{\sigma_2} j_{\sigma_3}} \ldots \delta_{i_{\sigma_n} j_{\sigma_1}} A_n\left(\sigma_1, \ldots, \sigma_n\right), \quad (2.1)$$

where the sum is over all non-cyclic permutations of the external gluon legs. The quantities $A_n(\sigma_1, \ldots, \sigma_n)$, called the partial amplitudes, contain the kinematic information. They are colour-ordered, e.g. only diagrams with a particular cyclic ordering of the gluons contribute. The choice of the basis for the colour structures is not unique, and several proposals for bases can be found in the literature [49, 50]. Here we use the "colour-flow decomposition" [50, 51]. As a further example we give the the colour decomposition for a tree amplitude with a pair of quarks:

$$\mathcal{A}_{n+2}(q, 1, 2, \ldots, n, \bar{q}) = \left(\frac{g}{\sqrt{2}}\right)^n \sum_{S_n} \delta_{i_q j_{\sigma_1}} \delta_{i_{\sigma_1} j_{\sigma_2}} \ldots \delta_{i_{\sigma_n} j_{\bar{q}}} A_{n+2}(q, \sigma_1, \sigma_2, \ldots, \sigma_n, \bar{q}) (2.2)$$

where the sum is over all permutations of the gluon legs. In squaring these amplitudes a colour projector

$$\delta_{\bar{i}i} \delta_{j\bar{j}} - \frac{1}{N} \delta_{\bar{i}\bar{j}} \delta_{ji} \tag{2.3}$$

has to applied to each gluon. While the colour structures of the examples quoted above are rather simple, the colour decomposition can be become rather involved for amplitudes

with many pairs of quarks. A systematic algorithm for the colour decomposition and the diagrams contributing to a single colour structure can be found in ref. [51].

While not strictly necessary, we consider in this paper only colour-ordered partial amplitudes. These partial amplitudes are cyclic ordered within each colour cluster. The cyclic order reduces significantly the number of possibilities of dividing n external particles into two set, such that particle i belongs to one set, while particle j belongs to the other set.

2.2 Spinors and polarisation vectors

Let us consider two independent Weyl spinors $|q+\rangle$ and $\langle q+|$. These two Weyl spinors define a light-like four-vector

$$q^\mu = \frac{1}{2}\langle q+|\gamma^\mu|q+\rangle. \tag{2.4}$$

This four-vector can be used to associate to any not necessarily light-like four-vector k a light-like four-vector $k^\flat$:

$$k^\flat = k - \frac{k^2}{2k \cdot q}q. \tag{2.5}$$

The four-vector $k^\flat$ satisfies $(k^\flat)^2 = 0$. We can generalise this construction and associate to an arbitrary four-vector K a four-vector $K_m^\flat$ defined through

$$K_m^\flat = K - \frac{\left(K^2 - m^2\right)}{2K \cdot q}q, \tag{2.6}$$

which satisfies

$$\left(K_m^\flat\right)^2 = m^2. \tag{2.7}$$

Therefore $K_m^\flat$ corresponds to the momentum of an on-shell particle with mass m. It is worth noting that starting from K and constructing directly a light-like four-vector $K^\flat$ through eq. (2.5) is the same as first constructing $K_m^\flat$ by eq. (2.6) and then projecting $K_m^\flat$ onto a light-like vector $(K_m^\flat)^\flat$:

$$\left(K_m^\flat\right)^\flat = K^\flat. \tag{2.8}$$

The two Weyl spinors $|q+\rangle$ and $\langle q+|$ are also used as reference spinors in the definition of the polarisations of the external particles. For massive fermions we take the spinors as [52, 43]

$$u(-) = \frac{1}{\langle p^\flat + |q-\rangle}(\not{p}+m)|q-\rangle, \quad \bar{u}(+) = \frac{1}{\langle q-|p^\flat+\rangle}\langle q-|(\not{p}+m),$$

$$u(+) = \frac{1}{\langle p^\flat - |q+\rangle}(\not{p}+m)|q+\rangle, \quad \bar{u}(-) = \frac{1}{\langle q+|p^\flat-\rangle}\langle q+|(\not{p}+m). \tag{2.9}$$

These expressions are similar to the ones introduced in [53], the major difference is given by the fact, that the denominators contain spinor products rather than ordinary square roots.

The spinor $u(p)$ corresponds to a particle with incoming momentum, therefore it has the reversed helicity notation compared to the usual conventions [53, 14]. This notation will turn out to simplify the discussion of the allowed helicity combinations of shifted quark lines in section 3.5 since the same restrictions apply to outgoing quarks and incoming antiquarks. Furthermore, using the conventions (2.9) internal propagators of quarks in the BCFW relation connect $+$ and $-$ labels, as in the gluon case. We note for completeness that the spinors $v(\pm)$ and $\bar{v}(\pm)$ are given by

$$v(\pm) = \frac{1}{\langle p^\flat \mp |q\pm\rangle} (\not{p} - m) |q\pm\rangle, \quad \bar{v}(\pm) = \frac{1}{\langle q \mp |p^\flat\pm\rangle}\langle q \mp |(\not{p} - m). \qquad (2.10)$$

These spinors satisfy the Dirac equations

$$(\not{p} - m)\, u(\lambda) = 0, \quad \bar{u}(\lambda)\,(\not{p} - m) = 0, \qquad (2.11)$$

the orthogonality relations

$$\bar{u}(\bar{\lambda})u(-\lambda) = 2m\delta_{\bar{\lambda}\lambda}, \qquad (2.12)$$

and the completeness relation

$$\sum_\lambda u(-\lambda)\bar{u}(\lambda) = \not{p} + m. \qquad (2.13)$$

We further have

$$\bar{u}(\bar{\lambda})\gamma^\mu u(-\lambda) = 2p^\mu \delta_{\bar{\lambda}\lambda}. \qquad (2.14)$$

In the massless limit the definition reduces to

$$\begin{aligned} u(-) &= |p+\rangle, \quad \bar{u}(+) = \langle p + |, \\ u(+) &= |p-\rangle, \quad \bar{u}(-) = \langle p - |, \end{aligned} \qquad (2.15)$$

and the spinors are independent of the reference spinors $|q+\rangle$ and $\langle q+|$. For the polarisation vectors of the gluons we take

$$\varepsilon^+_\mu = \frac{\langle q - |\gamma_\mu|k-\rangle}{\sqrt{2}\langle q - |k+\rangle}, \quad \varepsilon^-_\mu = \frac{\langle q + |\gamma_\mu|k+\rangle}{\sqrt{2}\langle k + |q-\rangle}. \qquad (2.16)$$

The dependence on the reference spinors which enters through the gluon polarisation vectors will drop out in gauge invariant quantities. In addition, as we have seen, the external spinors of massless fermions are explicitly independent of the reference spinors. Therefore we find again that (gauge invariant) amplitudes will not depend on them. However for massive fermions the reference spinors are related to the quantisation axis of the spin for this fermion, and the individual amplitudes with label $+$ or $-$ will therefore depend on the reference spinors $|q+\rangle$ and $\langle q+|$. It is easy to relate helicity amplitudes of massive quarks corresponding to one choice of reference spinors to another set of reference spinors. If $|\tilde{q}+\rangle$ and $\langle \tilde{q}+|$ is a second pair of reference spinors we have the following transformation law

$$\begin{pmatrix} \bar{u}(+, \tilde{q}) \\ \bar{u}(-, \tilde{q}) \end{pmatrix} = \begin{pmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{pmatrix} \begin{pmatrix} \bar{u}(+, q) \\ \bar{u}(-, q) \end{pmatrix}, \qquad (2.17)$$

where

$$c_{11} = \frac{\langle \tilde{q} - |\not{p}|q-\rangle}{\langle \tilde{q}\tilde{p}^\flat\rangle[p^\flat q]}, \quad c_{12} = \frac{m\langle \tilde{q}q\rangle}{\langle \tilde{q}\tilde{p}^\flat\rangle\langle p^\flat q\rangle}, \quad c_{21} = \frac{m[\tilde{q}q]}{[\tilde{q}\tilde{p}^\flat][p^\flat q]}, \quad c_{22} = \frac{\langle \tilde{q} + |\not{p}|q+\rangle}{[\tilde{q}\tilde{p}^\flat]\langle p^\flat q\rangle}. \quad (2.18)$$

Here, $\tilde{p}^\flat$ denotes the projection onto a light-like four-vector with respect to the reference vector $\frac{1}{2}\langle \tilde{q} + |\gamma^\mu|\tilde{q}+\rangle$. Similar, we have for an amplitude with an incoming massive quark

$$\begin{pmatrix} u(+, \tilde{q}) \\ u(-, \tilde{q}) \end{pmatrix} = \begin{pmatrix} c_{11} & -c_{12} \\ -c_{21} & c_{22} \end{pmatrix} \begin{pmatrix} u(+, q) \\ u(-, q) \end{pmatrix}. \quad (2.19)$$

3. The recursion relation

In this section we state the on-shell recursion relation for Born QCD amplitudes. Conventionally, an amplitude depends on a set of external momenta $\{p_1, p_2, \ldots, p_n\}$. In a first step we replace each four-vector by two spinors and view a QCD amplitude as a function of these spinors. In 3.1 we show how to recover the original four-vectors from the spinors. The recursion relation shifts the spinors by massless spinors. Since we allow for massive external particles, we have to associate to a pair of two external particles two pairs of massless spinors. A convenient Lorentz-invariant solution is given in 3.2. With these spinors at hand we state the holomorphic shift and the anti-holomorphic shift in 3.3 and 3.4, respectively. Finally, 3.5 assembles all ingredients and gives the recursion relation. Here we also present a list of the allowed helicity combinations. The proof of the recursion relation is deferred to section 4.

3.1 Arguments of the amplitudes

To state the recursion relation it is best not to view a QCD amplitude as a function of a set of four-momenta $\{p_1, p_2, \ldots, p_n\}$, but to replace each four-vector p_j by two spinors $u_j(-)$ and $\bar{u}_j(+)$. It is sufficient to specify these two spinors, since the remaining spinors $u_j(+)$ and $\bar{u}_j(-)$ can be obtained by raising and lowering dotted or undotted indices. If we change for the moment from the bra-ket notation to the one with dotted/undotted indices according to

$$|p+\rangle = p_A, \quad \langle p + | = p_{\dot{A}}, \quad (3.1)$$

$$|p-\rangle = p^{\dot{A}}, \quad \langle p - | = p^A, \quad (3.2)$$

we have

$$u(-) = p_A^\flat + \frac{m}{\langle p^\flat + |q-\rangle}q^{\dot{B}}, \quad \bar{u}(+) = p_{\dot{A}}^\flat + \frac{m}{\langle q - |p^\flat+\rangle}q^B. \quad (3.3)$$

$u(+)$ and $\bar{u}(-)$ are then given by

$$\bar{u}(-) = p^{\flat A} + \frac{m}{\langle q + |p^\flat-\rangle}q_{\dot{B}}, \quad u(+) = p^{\flat \dot{A}} + \frac{m}{\langle p^\flat - |q+\rangle}q_B, \quad (3.4)$$

where $p^{\flat A}$, $p^{\flat \dot{A}}$, $q_{\dot{B}}$ and q_B are obtained as

$$p^{\flat A} = \varepsilon^{AB}p_B^\flat, \quad p^{\flat \dot{A}} = \varepsilon^{\dot{A}\dot{B}}p_{\dot{B}}^\flat, \quad q_{\dot{B}} = q^{\dot{A}}\varepsilon_{\dot{A}\dot{B}}, \quad q_B = q^A\varepsilon_{AB}. \quad (3.5)$$

The two-dimensional antisymmetric tensor is defined by

$$\varepsilon^{AB} = \varepsilon^{\dot{A}\dot{B}} = \varepsilon_{AB} = \varepsilon_{\dot{A}\dot{B}} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \tag{3.6}$$

We see that $u(-)$ determines $\bar{u}(-)$, and that correspondingly $\bar{u}(+)$ determines $u(+)$. Given the spinors we obtain the four-vector p^μ as follows:

$$p^\mu = \frac{1}{4} \operatorname{Tr} \left(\gamma^\mu \sum_\lambda u(-\lambda)\bar{u}(\lambda) \right). \tag{3.7}$$

Eq. (3.7) in combination with eq. (3.4) allows the reconstruction of each four-vector p_j^μ from the two spinors $u_j(-)$ and $\bar{u}_j(+)$.

3.2 Choosing the spinors

To derive the recursion relation we mark two particles i and j, which need not be massless, with four-momenta p_i and p_j. To these two four-momenta we associate two light-like four-momenta l_i and l_j as follows [54, 55]: If p_i and p_j are massless, l_i and l_j are given by

$$l_i = p_i, \quad l_j = p_j. \tag{3.8}$$

If p_i is massless, but p_j is massive one has

$$l_i = p_i, \quad l_j = -\alpha_i p_i + p_j, \quad \alpha_i = \frac{p_j^2}{2p_i p_j}. \tag{3.9}$$

The inverse formula is given by

$$p_i = l_i, \quad p_j = \alpha_i l_i + l_j. \tag{3.10}$$

If both p_i and p_j are massive, one has

$$l_i = \frac{1}{1 - \alpha_i \alpha_j} \left(p_i - \alpha_j p_j \right), \quad l_j = \frac{1}{1 - \alpha_i \alpha_j} \left(-\alpha_i p_i + p_j \right). \tag{3.11}$$

α_1 and α_2 are given by

$$\alpha_j = \frac{2p_i p_j - \operatorname{sign}(2p_i p_j)\sqrt{\Delta}}{2p_j^2}, \quad \alpha_i = \frac{2p_i p_j - \operatorname{sign}(2p_i p_j)\sqrt{\Delta}}{2p_i^2}. \tag{3.12}$$

Here,

$$\Delta = (2p_i p_j)^2 - 4p_i^2 p_j^2. \tag{3.13}$$

The signs are chosen in such a way that the massless limit $p_i^2 \rightarrow 0$ (or $p_j^2 \rightarrow 0$) are approached smoothly. The inverse formula is given by

$$p_i = l_i + \alpha_j l_j, \quad p_j = \alpha_i l_i + l_j. \tag{3.14}$$

Note that l_1, l_2 are real for $\Delta > 0$. For $\Delta < 0$, l_1 and l_2 acquire imaginary parts. As a summary we can associate to any pair (p_i, p_j) of four-vectors a pair of light-like four-vectors (l_i, l_j). These light-like four-vectors define massless spinors $|l_i+\rangle$, $\langle l_i + |$, $|l_j+\rangle$ and $\langle l_j + |$. Explicit formulae for the construction of these spinors are given in appendix A.

3.3 The holomorphic shift

In an amplitude we single out two particles (massive or not) for special treatment. From the four-vectors p_i and p_j we first obtain the two light-like four-vectors l_i and l_j and the associated massless spinors $|l_i+\rangle$, $\langle l_i + |$, $|l_j+\rangle$ and $\langle l_j + |$. We consider helicity amplitudes. If particle i is a massive quark or anti-quark, we use $|l_j+\rangle$ and $\langle l_j + |$ as reference spinors for particle i. If particle j is a massive quark or anti-quark, we use $|l_i+\rangle$ and $\langle l_i + |$ as reference spinors for particle j. In this case it is an easy exercise to show that

$$p_i^\flat = l_i, \ \ p_j^\flat = l_j. \tag{3.15}$$

For massive particles the reference momenta define the spin quantisation axis. If particle i or j is a gluon, we leave the corresponding reference spinors unspecified. Gauge invariant quantities do not depend on the choice of reference spinors for gluons. In the rest of the paper we will often choose specific reference spinors. It should be understood that this choice only affects massive quarks or anti-quarks. The spinors read in detail:

$$u_i(-) = |l_i+\rangle + \frac{m_i}{[l_i l_j]}|l_j-\rangle, \ \ \bar{u}_i(+) = \langle l_i + | + \frac{m_i}{\langle l_j l_i\rangle}\langle l_j - |,$$
$$u_j(-) = |l_j+\rangle + \frac{m_j}{[l_j l_i]}|l_i-\rangle, \ \ \bar{u}_j(+) = \langle l_j + | + \frac{m_j}{\langle l_i l_j\rangle}\langle l_i - |. \tag{3.16}$$

For the holomorphic shift we shift $u_i(-)$ and $\bar{u}_j(+)$, while $u_j(-)$ and $\bar{u}_i(+)$ remain unchanged:

$$u_i'(-) = u_i(-) - z|l_j+\rangle, \ \ \bar{u}_i'(+) = \bar{u}_i(+),$$
$$u_j'(-) = u_j(-), \ \ \ \ \ \ \ \ \ \ \ \ \bar{u}_j'(+) = \bar{u}_j(+) + z\langle l_i + |. \tag{3.17}$$

If both particles are massless we have $l_i = p_i$ and $l_j = p_j$. Then the shift defined in eq. (3.17) reduces to the well-known form

$$|p_i'+\rangle = |p_i+\rangle - z|p_j+\rangle, \ \ \langle p_i' + | = \langle p_i + |,$$
$$|p_j'+\rangle = |p_j+\rangle, \ \ \ \ \ \ \ \ \ \ \ \ \langle p_j' + | = \langle p_j + | + z\langle p_i + |. \tag{3.18}$$

The spinors $u_i'(-)$ and $\bar{u}_i'(+)$ correspond to an on-shell particle with mass m_i and four-momentum

$$p_i'^\mu = p_i^\mu - \frac{z}{2}\langle l_i + |\gamma^\mu| l_j+\rangle. \tag{3.19}$$

The spinors $u_j'(-)$ and $\bar{u}_j'(+)$ correspond to an on-shell particle with mass m_j and four-momentum

$$p_j'^\mu = p_j^\mu + \frac{z}{2}\langle l_i + |\gamma^\mu| l_j+\rangle. \tag{3.20}$$

It is worth to examine the requirement to use $|l_j+\rangle$ and $\langle l_j + |$ as reference spinors for particle i in detail. Assume that we have an arbitrary spin quantisation axis described by the reference spinors $|q+\rangle$ and $\langle q + |$. As before we perform the shift

$$u_i'(-) = u_i(-) - z|l_j+\rangle, \ \ \bar{u}_i'(+) = \bar{u}_i(+). \tag{3.21}$$

If we now consider the polarisation sum we find

$$\sum_\lambda u_i'(-\lambda)\bar{u}_i'(\lambda) = \not{p}_i + m_i - z \begin{pmatrix} \frac{m_i}{\langle p_i^\flat q\rangle}(|q+\rangle\langle l_j - | - |l_j+\rangle\langle q - |) & |l_j+\rangle\langle p_i^\flat + | \\ |p_i^\flat-\rangle\langle l_j - | & 0 \end{pmatrix}. \quad (3.22)$$

As this polarisation sum must have the form

$$\not{p}_i' + m_i, \quad (3.23)$$

we have to require that the entry in the upper left corner vanishes:

$$|q+\rangle\langle l_j - | \ - \ |l_j+\rangle\langle q - | = 0. \quad (3.24)$$

Therefore it follows that $|q+\rangle = \lambda|l_j+\rangle$. The requirement $\langle q + | = \lambda'\langle l_j + |$ follows from similar considerations related to the anti-holomorphic shift discussed in the next sub-section. Because not all helicity combinations can be computed with the holomorphic shift, we have to fix both reference spinors $|q+\rangle$ and $\langle q + |$, and use the anti-holomorphic shift as well as the holomorphic shift to compute all helicity combinations. This allows us to recover the helicity amplitudes for arbitrary reference spinors from (2.17) and (2.19) .

Finally we remark that the spinors should not lead to spurious poles in z in the analytically continued scattering amplitude $A(z)$. This excludes for instance the choice $u_i'(-) = (\not{p}_i' + m)|q-\rangle/[p_i^\flat q]$ and $\bar{u}_i'(+) = \langle q - |(\not{p}_i' + m)/(\langle q p_i^\flat\rangle - z\langle q l_j\rangle)$ for $|q+\rangle \neq |l_j+\rangle$.

Let k be an intermediate particle where we would like to factorise the amplitude. We denote by K the off-shell four-momentum flowing through this propagator in the unshifted amplitude. We define the polarisations with respect to the reference spinors $|l_j+\rangle$ and $\langle l_i + |$:

$$u_K'(-) = \frac{1}{\langle K^\flat + |l_i-\rangle} \left(\not{K}' + m_k \right) |l_i-\rangle,$$
$$\bar{u}_K'(+) = \frac{1}{\langle l_j - |K^\flat+\rangle} \langle l_j-| \left(\not{K}' + m_k \right), \quad (3.25)$$

where

$$K'^\mu = K^\mu - \frac{z}{2}\langle l_i + |\gamma^\mu|l_j+\rangle, \quad K^{\flat\mu} = K^\mu - \frac{1}{2}\frac{K^2}{\langle l_i + |K|l_j+\rangle}\langle l_i + |\gamma^\mu|l_j+\rangle. \quad (3.26)$$

$K^\flat$ is a light-like four-vector. Note that $K^\flat = (K')^\flat$. Furthermore K' is on-shell $((K')^2 = m_k^2)$ provided

$$z = \frac{K^2 - m_k^2}{\langle l_i + |K|l_j+\rangle}. \quad (3.27)$$

3.4 The anti-holomorphic shift

For the anti-holomorphic shift we modify $\bar{u}_i(+)$ and $u_j(-)$:

$$\begin{aligned} u_i'(-) &= u_i(-), & \bar{u}_i'(+) &= \bar{u}_i(+) - z\langle l_j + |, \\ u_j'(-) &= u_j(-) + z|l_i+\rangle, & \bar{u}_j'(+) &= \bar{u}_j(+). \end{aligned} \quad (3.28)$$

If both particles are massless the shift defined in eq. (3.28) reduces to the form

$$
\begin{aligned}
|p_i'+\rangle &= |p_i+\rangle, & \langle p_i'+| &= \langle p_i+| - z\langle p_j+|, \\
|p_j'+\rangle &= |p_j+\rangle + z|p_i+\rangle, & \langle p_j'+| &= \langle p_j+|.
\end{aligned}
\tag{3.29}
$$

The spinors $u_i{}'(-)$ and $\bar{u}_i'(+)$ correspond to an on-shell particle with mass m_i and four-momentum

$$
p_i'{}^{\mu} = p_i^{\mu} - \frac{z}{2} \langle l_j + |\gamma^{\mu}| l_i+\rangle.
\tag{3.30}
$$

The spinors $u_j{}'(-)$ and $\bar{u}_j'(+)$ correspond to an on-shell particle with mass m_j and four-momentum

$$
p_j'{}^{\mu} = p_j^{\mu} + \frac{z}{2} \langle l_j + |\gamma^{\mu}| l_i+\rangle.
\tag{3.31}
$$

Again, let k be an intermediate particle with off-shell four-momentum K. We define the polarisations now with respect to the reference spinors $|l_i+\rangle$ and $\langle l_j + |$:

$$
u_K'(-) = \frac{1}{\langle K^{\flat} + |l_j-\rangle} \left(\slashed{K}' + m_k \right) |l_j-\rangle, \quad \bar{u}_K'(+) = \frac{1}{\langle l_i - |K^{\flat}+\rangle} \langle l_i-| \left(\slashed{K}' + m_k \right),
\tag{3.32}
$$

where

$$
K'^{\mu} = K^{\mu} - \frac{z}{2}\langle l_j + |\gamma^{\mu}|l_i+\rangle, \quad K^{\flat\mu} = K^{\mu} - \frac{1}{2}\frac{K^2}{\langle l_j + |K|l_i+\rangle}\langle l_j + |\gamma^{\mu}|l_i+\rangle.
\tag{3.33}
$$

$K^{\flat}$ is a light-like four-vector and we have $K^{\flat} = (K')^{\flat}$. Furthermore K' is on-shell $((K')^2 = m_k^2)$ provided

$$
z = \frac{K^2 - m_k^2}{\langle l_j + |K|l_i+\rangle}.
\tag{3.34}
$$

3.5 Assembling the ingredients: the recursion relation

We can now state the recursion relation. The starting point is the function

$$
A(z) = A_n \left(u_1(-), \bar{u}_1(+), \lambda_1, \ldots, u_i'(-), \bar{u}_i'(+), \lambda_i, \ldots, u_j'(-), \bar{u}_j'(+), \lambda_j, \ldots \right),
\tag{3.35}
$$

where the spinors of particles i and j have been shifted either with the holomorphic shift or with the anti-holomorphic shift. The amplitude we want to calculate is given by $A(0)$.

If the shifted amplitude $A(z)$ vanishes for $z \to \infty$ we obtain:

$$
A_n \left(u_1(-), \bar{u}_1(+), \lambda_1, \ldots, u_n(-), \bar{u}_n(+), \lambda_n \right) =
\tag{3.36}
$$

$$
\sum_{\text{partitions}} \sum_{\lambda=\pm} A_L \left(\ldots, u_i'(-), \bar{u}_i'(+), \lambda_i, \ldots, iv_K'(-), i\bar{v}_K'(+), -\lambda \right)
$$

$$
\times \frac{i}{K^2 - m_k^2} A_R \left(u_K'(-), \bar{u}_K'(+), \lambda, \ldots, u_j'(-), \bar{u}_j'(+), \lambda_j, \ldots \right),
$$

where the sum is over all partitions such that particle i is on the left and particle j is on the right. The momentum K is given as the sum over all unshifted momenta of the original external particles, which are part of A_L. The values of z are given for the holomorphic shift by eq. (3.27) and for the anti-holomorphic shift by eq. (3.34).

The condition that $A(z)$ has to vanish at infinity can be summarised as follows:

- Particles i and j cannot belong to the same fermion line.

- The holomorphic shift can be used for the helicity combinations (i^+, j^-), (i^+, j^+) and (i^-, j^-) with the following exceptions:

  - The combinations (q_i^+, g_j^+), $(\bar{q}_i^+, g_j^+)$, (g_i^-, q_j^-) and $(g_i^-, \bar{q}_j^-)$ are not allowed.
  - If particle i is a massive quark or anti-quark, the combinations $(q_i^+, q_j'^+)$, $(q_i^+, \bar{q}_j'^+)$, $(\bar{q}_i^+, q_j'^+)$ and $(\bar{q}_i^+, \bar{q}_j'^+)$ are not allowed.
  - If particle j is a massive quark or anti-quark, the combinations $(q_i^-, q_j'^-)$, $(q_i^-, \bar{q}_j'^-)$, $(\bar{q}_i^-, q_j'^-)$ and $(\bar{q}_i^-, \bar{q}_j'^-)$ are not allowed.

- The anti-holomorphic shift can be used for the helicity combinations (i^-, j^+), (i^+, j^+) and (i^-, j^-) with the following exceptions:

  - The combinations (g_i^+, q_j^+), $(g_i^+, \bar{q}_j^+)$, (q_i^-, g_j^-) and $(\bar{q}_i^-, g_j^-)$ are not allowed.
  - If particle j is a massive quark or anti-quark, the combinations $(q_i^+, q_j'^+)$, $(q_i^+, \bar{q}_j'^+)$, $(\bar{q}_i^+, q_j'^+)$ and $(\bar{q}_i^+, \bar{q}_j'^+)$ are not allowed.
  - If particle i is a massive quark or anti-quark, the combinations $(q_i^-, q_j'^-)$, $(q_i^-, \bar{q}_j'^-)$, $(\bar{q}_i^-, q_j'^-)$ and $(\bar{q}_i^-, \bar{q}_j'^-)$ are not allowed.

In summary there is always at least one allowed shift, unless i and j belong to the same fermion line or i and j are both massive quarks or anti-quarks. As we are free to choose the particles i and j, we can compute all Born helicity amplitudes in QCD with two-particle shifts via recursion relations, except the ones which involve only massive quarks or anti-quarks. The latter ones may be calculated recursively if one allows more general shifts, where more than two particles are shifted. This follows directly from the proof of the recursion relation which we present in section 4. Amplitudes with only massive quarks or anti-quarks are discussed in detail in section 4.3.

4. Proof of the recursion relation

The standard proof of the BCFW recursion relation is based on Cauchy's theorem [5]. The function $A(z)$ is a rational function of z, which has only simple poles in z. Therefore, if $A(z)$ vanishes for $z \to \infty$, $A(z)$ is given by Cauchy's theorem as the sum over its residues. This is just the right hand side of the recursion relation. The essential ingredient for the proof is the vanishing of $A(z)$ at $z \to \infty$. This property we have to verify for the shifts stated in the previous section.

	g_i^+	g_i^-	$\bar{Q}_i^+$	$\bar{Q}_i^-$	Q_i^+	Q_i^-	g_j^+	g_j^-	$\bar{Q}_j^+$	$\bar{Q}_j^-$	Q_j^+	Q_j^-
holomorphic	$\frac{1}{z}$	z	1	z	1	z	z	$\frac{1}{z}$	z	1	z	1
anti-holomorphic	z	$\frac{1}{z}$	z	1	z	1	$\frac{1}{z}$	z	1	z	1	z

Table 1: Behaviour of polarisation vectors and spinors in the large z limit for the holomorphic and anti-holomorphic shift.

It is relatively easy to show this for the helicity combination (i^+, j^-) for the holomorphic shift and for the helicity combination (i^-, j^+) for the anti-holomorphic shift. We do this in section 4.1.

The helicity combinations (i^+, j^+) and (i^-, j^-) require a more sophisticated proof. In section 4.2 we first construct a representation of $A(z)$ with the help of a supplementary recursion relation and deduct from this representation the large z-behaviour of $A(z)$. For the proof we borrowed ideas from [9, 20, 29, 36]. The proof presented here does not rely on additional (unnecessary) assumptions like the presence of two additional gluons with specific helicities.

4.1 Diagrammatic analysis of the large z behaviour

We now investigate the behaviour of $A(z)$ for large z by a diagrammatic analysis. A gluon propagator behaves like $1/z$, whereas a quark propagator tends towards a constant. The quark-gluon and the four-gluon vertices are independent of z, whereas the three-gluon vertex is proportional to z for large z.

The behaviour of the polarisation vectors and spinors are summarised in table 1.

As a first observation we note that a shift of two quarks belonging to the same fermion line is not allowed. In all diagrams the z-dependence flows along this fermion line, which consists of quark propagators and quark-gluon vertices. These tend towards a constant for large z. The large z behaviour of the external spinors tends towards a constant at the best. Therefore, we conclude that independent of the helicities the function $A(z)$ does not vanish for $z \to \infty$.

Let us now assume that the two shifted particles belong to different fermion lines, or that one or both particles are gluons. Therefore we have at least one gluon propagator along the shifted line, except for the case where the shifted line does not contain any propagators at all. By a diagrammatic analysis one can easily show that the helicity combination (i^+, j^-) behaves like $1/z$ for $z \to \infty$ for the holomorphic shift, independent of the nature of the particles i and j. The reversed helicity assignment (i^-, j^+) behaves like $1/z$ for the anti-holomorphic shift. To see this, let us consider as an example the holomorphic shift. Assume first that the flow of z-dependence in a particular diagram is given by a path made out entirely of gluons. The most dangerous contribution comes from a path, where all vertices are three-gluon-vertices. For a path made of n propagators we have $n + 1$ vertices and the product of propagators and vertices behaves therefore like z for large z. This

statement remains true for a path containing only one vertex and no propagators. The polarisation vectors for the helicity combination (i^+, j^-) contribute a factor $1/z^2$, therefore the complete diagram behaves like $1/z$ and vanishes therefore for $z \to \infty$. If internally a gluon propagator is replaced by a quark propagator, we have to change at least two three-gluon vertices into quark-gluon vertices. This improves the estimate by a factor $1/z$. If an external gluon is replaced by a fermion, we have to change at least one three-gluon vertex into a quark-gluon vertex. This does not modify the large z behaviour.

4.2 Supplementary recursion relation for the large z behaviour

The cases (i^+, j^+) and (i^-, j^-) are more subtle. As an example we consider the case (i^+, j^+) with the holomorphic shift. The other cases, (i^+, j^+) with the anti-holomorphic shift and (i^-, j^-) with the holomorphic as well as with the anti-holomorphic shift will be similar.

We are going to prove that in the case (i^+, j^+) and for the holomorphic shift the function $A(z)$ vanishes as $z \to \infty$. We prove this for the case where the spinors $u_i(-)$, $\bar{u}_i(+)$, $u_j(-)$ and $\bar{u}_j(+)$ are defined with respect to the reference spinors

$$|q_i+\rangle = |l_j+\rangle, \quad \langle q_j + | = \langle l_i + |. \tag{4.1}$$

Compared to section 3.3 we do not require any particular choice for the reference spinors $\langle q_i + |$ and $|q_j+\rangle$. We can write these last two reference spinors as linear combinations of two basis spinors, and since we are free to choose the normalisation of the reference spinors we can write them without loss of generality as

$$\langle q_i + | = \langle l_j + | + \lambda_i \langle l_i + |, \quad |q_j+\rangle = |l_i+\rangle + \lambda_j |l_j+\rangle, \tag{4.2}$$

where λ_i and λ_j are complex numbers. A simple calculation shows that we then obtain with these reference spinors

$$|p_i^\flat+\rangle = |l_i+\rangle - \lambda_i \frac{m_i^2}{2l_i l_j}|l_j+\rangle, \quad \langle p_i^\flat + | = \langle l_i + |,$$

$$|p_j^\flat+\rangle = |l_j+\rangle, \qquad\qquad \langle p_j^\flat + | = \langle l_j + | - \lambda_j \frac{m_j^2}{2l_i l_j}\langle l_i + |. \tag{4.3}$$

The spinors $u_i(-)$, $\bar{u}_i(+)$, $u_j(-)$ and $\bar{u}_j(+)$ read then

$$u_i(-) = |p_i^\flat+\rangle + \frac{m_i}{[l_i q_i]}|q_i-\rangle, \quad \bar{u}_i(+) = \langle l_i + | + \frac{m_i}{\langle l_j l_i\rangle}\langle l_j - |,$$

$$u_j(-) = |l_j+\rangle + \frac{m_j}{[l_j l_i]}|l_i-\rangle, \quad \bar{u}_j(+) = \langle p_j^\flat + | + \frac{m_j}{\langle q_j l_j\rangle}\langle q_j - |. \tag{4.4}$$

The holomorphic shift is chosen as in eq. (3.17):

$$u_i{'}(-) = u_i(-) - z|l_j+\rangle, \quad \bar{u}_i{'}(+) = \bar{u}_i(+),$$

$$u_j{'}(-) = u_j(-), \qquad\qquad \bar{u}_j{'}(+) = \bar{u}_j(+) + z\langle l_i + |. \tag{4.5}$$

We give a proof by induction in the number of external particles.

We first show that the three-point functions vanish for $z \to \infty$. We start with the pure gluon case. $A_3(g_i'^+, g_j'^+, g_k^+)$ vanishes identically, whereas $A_3(g_i'^+, g_j'^+, g_k^-)$ as a function of z is given by

$$A_3(g_i'^+, g_j'^+, g_k^-) = i\sqrt{2} \frac{[ji]^3}{[ki]\left([jk] + z[ik]\right)}. \tag{4.6}$$

Clearly, this function vanishes for $z \to \infty$.

Let us now consider the case, where particle i is a gluon and particle j is a quark. Then the third particle k is necessarily an anti-quark. For a massive fermion line we have to consider both helicities for particle k. A short calculation shows that with the choice of reference spinors as in eq. (4.1) we have

$$
\begin{aligned}
A_3(g_i'^+, Q_j'^+, \bar{Q}_k^-) &= A_3(g_i'^+, \bar{Q}_j'^+, Q_k^-) = 0, \\
A_3(g_i'^+, Q_j'^+, \bar{Q}_k^+) &= A_3(g_i'^+, \bar{Q}_j'^+, Q_k^+) = 0.
\end{aligned} \tag{4.7}
$$

These amplitudes certainly vanish for $z \to \infty$. On the other it can be shown that the amplitudes $A_3(\bar{Q}_i'^+, g_j'^+, Q_k^-)$ and $A_3(Q_i'^+, g_j'^+, \bar{Q}_k^-)$ do not vanish for $z \to \infty$. If the quark is massive, the same holds for the amplitudes $A_3(\bar{Q}_i'^+, g_j'^+, Q_k^+)$ and $A_3(Q_i'^+, g_j'^+, \bar{Q}_k^+)$. This places the constraint that if particle i is a quark or an anti-quark, particle j is a gluon and the two are adjacent, then the helicity combination (i^+, j^+) cannot be calculated with the holomorphic shift.

There are a few 4- and 5-point amplitudes, which we treat separately:

$$A_4(g_i^+, g_j^+, Q, Q), \quad A_4(Q_i^+, g_j^+, Q, g), \quad A_5(Q_i^+, g_j^+, Q, Q', Q'), \quad A_4(Q_i^+, Q_j'^+, Q, Q'). \tag{4.8}$$

Here Q and Q' stands either for a quark or an anti-quark and no particular cyclic order is implied. The amplitudes in eq. (4.8) are the only four- or higher-point amplitudes, where we cannot choose in addition to the marked particles i and j two additional particles k and l such that in the set

$$\{i, k, l\} \tag{4.9}$$

no fermion line connects two of the three external particles. These cases are discussed in appendix B and give rise to the following constraints: The holomorphic shift cannot be used for the combination (Q_i^+, g_j^+). For the combination $(Q_i^+, Q_j'^+)$ the holomorphic shift can only be used if $m_i = 0$.

We now proceed by induction in the number of external particles. We can assume that there are two additional particles k and l. Since we excluded the special cases in eq. (4.8), we can also assume that in the set $\{i, k, l\}$ no two particles belong to the same fermion line. We first discuss the case, where we can choose the two additional particles with identical helicities. These are the sub-cases

$$(i^+, j^+, k^-, l^-), \quad \text{and} \quad (i^+, j^+, k^+, l^+). \tag{4.10}$$

In these cases we first consider a supplementary recursion relation, which will provide us with an expression of the amplitude from which we can deduce the large z behaviour. This

leaves the sub-case, where we cannot choose two additional particles with equal helicity assignments. In this case k and l have opposite helicities and after a possible relabelling $k \leftrightarrow l$ we can assume that the helicity assignment is (i^+, j^+, k^-, l^+). We discuss this sub-case separately.

We first consider the case (i^+, j^+, k^-, l^-). As above we fix as reference spinors $|q_i+\rangle = |l_j+\rangle$ and $\langle q_j + | = \langle l_i + |$, while $\langle q_i + |$ and $|q_j+\rangle$ are arbitrary. For particles k and l we choose as reference spinors

$$|q_k+\rangle = |q_l+\rangle = |l_j+\rangle, \quad \langle q_k + | = \langle q_l + | = \langle l_i + |. \tag{4.11}$$

This choice defines $p_k^\flat$ and $p_l^\flat$. We now consider the shift

$$\begin{aligned}
u_i'(-) &= u_i(-) - z|l_j+\rangle - y\beta_k|p_k^\flat+\rangle - y\beta_l|p_l^\flat+\rangle, \\
\bar{u}_j'(+) &= \bar{u}_j(+) + z\langle l_i + |, \\
\bar{u}_k'(+) &= \bar{u}_k(+) + y\beta_k\langle l_i + |, \\
\bar{u}_l'(+) &= \bar{u}_l(+) + y\beta_l\langle l_i + |,
\end{aligned} \tag{4.12}$$

where β_k and β_l are chosen as

$$\beta_k = \frac{\langle p_l^\flat l_j\rangle}{\langle p_l^\flat p_k^\flat\rangle}, \quad \beta_l = \frac{\langle p_k^\flat l_j\rangle}{\langle p_k^\flat p_l^\flat\rangle}. \tag{4.13}$$

The coefficients are chosen such that

$$\beta_k|p_k^\flat+\rangle + \beta_l|p_l^\flat+\rangle = |l_j+\rangle. \tag{4.14}$$

The function $A(y)$ vanishes at infinity, as each individual diagram vanishes at infinity. The argument is similar to the one we gave above for the helicity combination (i^+, j^-): In any diagram the y-dependence flows through a three-legged path with end-point i, k and l. Suppose first that all three particles are gluons. The most dangerous diagrams are the ones, where we have only three-gluon-vertices along the path. Then the combination of propagators and vertices gives a factor y in the large y-behaviour, while the polarisation vectors contribute a factor $1/y^3$. In total this diagram goes like $1/y^2$ in the large y limit and therefore vanishes as y goes to infinity. If we replace internally a gluon propagator by a quark propagator, we have to change at least two three-gluon vertices into quark-gluon vertices. This improves the estimate by a factor $1/y$. If an external gluon is replaced by a fermion, we have to change at least one three-gluon vertex into a quark-gluon vertex. This does not modify the large y behaviour. Note that we have excluded the case, where a fermion line connects two of the three particles i, k and l.

From the fact that $A(y)$ vanishes for $y \to \infty$ we obtain the recursion relation

$$A(y = 0, z) = \sum_{\alpha, \lambda} A_L(y_\alpha, z, \lambda) \frac{i}{P_\alpha(z)^2 - m_\alpha^2} A_R(y_\alpha, z, -\lambda), \tag{4.15}$$

where we dropped arguments not relevant to the discussion here. We will use this formula to estimate the z-behaviour at infinity. Suppose that i and j are on opposite sides of the propagator. Then

$$P_\alpha(z)^2 = P_\alpha^2 - z\langle l_i + |\not{P}_\alpha|l_j+\rangle \tag{4.16}$$

and y_α depends linearly on z. If both k and l are on the same side as particle j we have

$$y_\alpha = \frac{P_\alpha^2 - m_\alpha^2 - z\langle l_i + |\not{P}_\alpha|l_j+\rangle}{\beta_k\langle l_i + |\not{P}_\alpha|p_k^\flat+\rangle + \beta_l\langle l_i + |\not{P}_\alpha|p_l^\flat+\rangle} = \frac{P_\alpha^2 - m_\alpha^2}{\langle l_i + |\not{P}_\alpha|l_j+\rangle} - z. \quad (4.17)$$

A similar formula holds if only one of the particles k or l is on the same side as particle j. The z-dependence flows through a four-legged path and one can show by a diagrammatic analysis that each diagram vanishes for $z \to \infty$. We first observe that the product of the scalar propagator, on which the amplitude is factorised, times the two polarisation vectors attached to it, behaves like an internal propagator in the large z limit. Let us again start from the pure gluon case and assume the worst-case scenario of only three-gluon vertices. The product of propagators and vertices gives a factor z, the three polarisation vectors for particles i, k and l contribute a factor $1/z^3$, the polarisation vector for particle j a factor z. In total the amplitude behaves like $1/z$ and vanishes in the large z limit. Replacing an internal gluon propagator by a quark propagator improves the estimate by a factor $1/z$. Replacing an external gluon by a quark does not change the large z-behaviour, as long as we do not have a fermion line connecting two of the four external particles i, j, k and l. As above, the cases where a fermion line connects two of the three particles i, k and l are excluded. In addition we have excluded from the very beginning the case where a fermion line connects i and j. Therefore the only possibilities, where a fermion line connects two particles are the ones where a fermion line connects particle j either with particle k or l. In this case the total contribution from this fermion line behaves like z for $z \to \infty$, while the rest of the diagram gives at least a factor $1/z^2$.

Suppose now that particles i and j are on the same side of the propagator, say they are both in A_L. Then y_α is independent of z. The reference spinors for particle i are given by $|l_j+\rangle$ and an arbitrary $\langle q_i+|$. For particle j the reference spinors are $|q_j+\rangle$ and $\langle l_i+|$. Since A_L has fewer legs than A we can use the induction hypothesis and therefore A_L vanishes as z goes to infinity. This completes the proof for the case (i^+, j^+, k^-, l^-).

We now consider the case (i^+, j^+, k^+, l^+). As reference spinors for particles k and l we take as above

$$|q_k+\rangle = |q_l+\rangle = |l_j+\rangle, \quad \langle q_k + | = \langle q_l + | = \langle l_i + |. \quad (4.18)$$

We consider the shift

$$\begin{aligned}
u_i'(-) &= u_i(-) - z|l_j+\rangle + y[p_k^\flat p_l^\flat]|l_j+\rangle, \\
\bar{u}_j'(+) &= \bar{u}_j(+) + z\langle l_i + |, \\
u_k'(-) &= u_k(-) + y[p_l^\flat l_i]|l_j+\rangle, \\
u_l'(-) &= u_l(-) + y[l_i p_k^\flat]|l_j+\rangle.
\end{aligned} \quad (4.19)$$

Momentum conservation is satisfied due to the Schouten identity. The shift in y is chosen such that each individual diagram vanishes for $y \to \infty$. Again we can show with the same steps as in the (i^+, j^+, k^-, l^-) case that $A(z)$ vanishes for $z \to \infty$.

Finally, we discuss the case (i^+, j^+, k^-, l^+). Assume first that particles i and j are massless particles. Then the amplitude is independent of the choice of the reference spinors

for particles i and j. As reference spinors for particles k and l we take again

$$|q_k+\rangle = |q_l+\rangle = |l_j+\rangle, \quad \langle q_k + | = \langle q_l + | = \langle l_i + |. \tag{4.20}$$

We consider the shift

$$\begin{aligned}
u'_i(-) &= u_i(-) - z|l_j+\rangle - y|p_k^\flat+\rangle, \\
\bar{u}'_j(+) &= \bar{u}_j(+) + z\langle l_i + | - y\langle p_l^\flat + |, \\
\bar{u}'_k(+) &= \bar{u}_k(+) + y\langle l_i + |, \\
u'_l(-) &= u_l(-) + y|l_j+\rangle.
\end{aligned} \tag{4.21}$$

We can show with the same steps as in the (i^+, j^+, k^-, l^-) case that $A(z)$ vanishes for $z \to \infty$. Note that for particle i and j the shift in y is not proportional to the reference spinors of these particles. Therefore the shift in eq. (4.21) is restricted to massless particles. This leaves the cases, where particle i or particle j or both are massive particles. In accordance with eq. (4.9) particles k and l are chosen such that in the set $\{i, k, l\}$ no fermion line connects two of the three external particles. There are only very few cases where k and l must be chosen such that they have opposite helicities. These are the cases

$$A_4(g_i^+, Q_j^+, Q^\pm, g^\mp), \quad A_5(q_i^+, Q_j'^+, q^-, Q'^\pm, g^\mp), \quad A_6(q_i^+, Q_j'^+, q^-, Q'^\pm, Q''^\mp, Q''^\mp). \tag{4.22}$$

Here q_i^+ denotes a massless quark, since the combination $(Q_i^+, Q_j'^+)$ where Q_i^+ is a massive quark is already excluded. All cases are discussed explicitly in appendix B. It will turn out that these cases do not lead to additional restriction on the validity of the recursion relation.

4.3 Amplitudes involving only massive quarks or anti-quarks

The holomorphic and anti-holomorphic two-particle shifts in eq. (3.17) and eq. (3.28) allow us to calculate recursively all amplitudes except the ones, which consist solely of massive quarks or anti-quarks. Among those, the four-parton amplitudes $A_4(\bar{Q}, Q, \bar{Q}', Q')$ are given by just one Feynman diagram and therefore are most efficiently calculated by a Feynman diagram calculation. Also the six-quark amplitudes are relatively simple.

We consider now the ones with more than six particles. We select two particles i and j, not belonging to the same fermion line. As reference spinors for particle i we choose

$$|q_i+\rangle = |l_j+\rangle, \quad \langle q_i + | = \langle l_j + |, \tag{4.23}$$

while for particle j we choose

$$|q_j+\rangle = |l_i+\rangle, \quad \langle q_j + | = \langle l_i + |. \tag{4.24}$$

For all other particles we choose as reference spinors

$$|q_k+\rangle = |l_j+\rangle, \quad \langle q_k + | = \langle l_i + |. \tag{4.25}$$

The helicity combination (i^+, j^-) can be calculated with the holomorphic shift eq. (3.17), while the combination (i^-, j^+) can be calculated with the anti-holomorphic shift eq. (3.28).

This leaves the combinations (i^+, j^+) and (i^-, j^-). We consider first the combination (i^+, j^+). As we are considering amplitudes with at least eight external particles, we can always find two particles k and l, such that in the set $\{i, k, l\}$ no fermion line connects two of the three particles and that in addition the particles k and l have the same helicity assignment. For the helicity combination (i^+, j^+, k^-, l^-) we can use the shift

$$
\begin{aligned}
u_i'(-) &= u_i(-) - z\beta_k |p_k^\flat +\rangle - z\beta_l |p_l^\flat +\rangle, \\
\bar{u}_k'(+) &= \bar{u}_k(+) + z\beta_k \langle l_i + |, \\
\bar{u}_l'(+) &= \bar{u}_l(+) + z\beta_l \langle l_i + |,
\end{aligned}
\tag{4.26}
$$

with

$$
\beta_k = \frac{\langle p_l^\flat l_j \rangle}{\langle p_l^\flat p_k^\flat \rangle}, \quad \beta_l = \frac{\langle p_k^\flat l_j \rangle}{\langle p_k^\flat p_l^\flat \rangle}.
\tag{4.27}
$$

This is just the three-particle shift we used to establish the supplementary recursion relation in section 4.2. For the helicity combination (i^+, j^+, k^+, l^+) we can use the shift

$$
\begin{aligned}
u_i'(-) &= u_i(-) + z[p_k^\flat p_l^\flat] |l_j +\rangle, \\
u_k'(-) &= u_k(-) + z[p_l^\flat l_i] |l_j +\rangle, \\
u_l'(-) &= u_l(-) + z[l_i p_k^\flat] |l_j +\rangle.
\end{aligned}
\tag{4.28}
$$

Similar considerations apply to the helicity combination (i^-, j^-).

5. Applications

In this section we present a few examples and applications. We discuss helicity amplitudes with a pair of massive quarks, zero or one negative helicity gluons and an arbitrary number of positive helicity gluons. Helicity amplitudes with a pair of massive quarks plus three gluons can be found in [56].

5.1 Amplitudes with positive helicity gluons

In this section we consider amplitudes with one massive quark pair and an arbitrary number of positive helicity gluons. These amplitudes are the building blocks for the construction of amplitudes with negative gluons using on-shell recursion relations. While the amplitudes for a pair of massive scalars or quarks and an arbitrary number of positive helicity gluons are known in closed form [11, 43, 42, 13], they serve as a first example to demonstrate the application of the shifts of momenta of massive quarks. Previous calculations of such amplitudes considered the shift of two gluons.

If the same spin axis is chosen for the two quarks, there are three non-vanishing amplitudes [52]: the helicity conserving amplitudes $A_n(Q_1^\pm, g_2^+, \ldots, \bar{Q}_n^\mp)$, and a helicity flip amplitude $A_n(Q_1^-, g_2^+, \ldots, \bar{Q}_n^+)$. The amplitude $A_n(Q_1^-, g_2^+, \ldots, \bar{Q}_n^+)$ is related by charge conjugation to the amplitude $A_n(Q_1^+, g_2^+, \ldots, \bar{Q}_n^-)$. As discussed in section 4 both the helicity conserving and the helicity flip amplitudes can be computed applying the holomorphic

shift (3.17) with $(i, j) = (2, 1)$. This implies that p_2 is chosen as reference momentum for Q_1 and $\bar{Q}_n$, but using the transformation (2.17) it is straightforward to obtain the results for an arbitrary polarisation. The recursion relation consists of a single term

$$A_n(Q_1^\pm, g_2^+, \ldots, \bar{Q}_n^-) = A_{n-1}(Q'_1{}^\pm, g'_{23}{}^+, g_4^+, \ldots, \bar{Q}_n^-) \frac{i}{p_{2,3}{}^2} A_3(g'_{(-23)}{}^-, g'_2{}^+, g_3^+), \quad (5.1)$$

with $p_{2,3} = p_2 + p_3$, since the degree zero amplitudes with more than three gluons vanish on-shell.

The amplitudes with a massive quark pair with the same spin quantisation axis are related through super-symmetric Ward identities to amplitudes of massive scalars [42]:

$$A_n(Q_1^+, g_2^+, \ldots, \bar{Q}_n^-) = \frac{\langle p_n^\flat q \rangle}{\langle p_1^\flat q \rangle} A_n(\phi_1^+, g_2^+, \ldots, \bar{\phi}_n^-), \quad (5.2)$$

$$A_n(Q_1^-, g_2^+, \ldots, \bar{Q}_n^-) = \frac{\langle p_1^\flat p_n^\flat \rangle}{m} A_n(\phi_1^+, g_2^+ \ldots, \bar{\phi}_n^-). \quad (5.3)$$

The scalar amplitudes satisfy therefore the recursion relation

$$A_n(\phi_1^+, g_2^+, \ldots, \bar{\phi}_n^-) = A_{n-1}(\phi'_1{}^+, g'_{23}{}^+, g_4^+, \ldots, \bar{\phi}_n^-) \frac{i}{p_{2,3}{}^2} A_3(g'_{(-23)}{}^-, g'_2{}^+, g_3^+). \quad (5.4)$$

The light-like momenta $p_1^\flat$ and $p_n^\flat$ associated to p_1 and p_n are given by

$$p_1^\flat = p_1 - \frac{m^2}{2p_1 p_2} p_2, \quad p_n^\flat = p_n - \frac{m^2}{2p_2 p_n} p_2. \quad (5.5)$$

The spinors are shifted as

$$|2+\rangle \to |2+\rangle - z|p_1^\flat+\rangle, \quad \bar{u}_1(+) \to \bar{u}_1(+) + z\langle 2+|, \quad (5.6)$$

where

$$z = \frac{p_{2,3}^2}{\langle 2+|\not{p}_3|p_1^\flat+\rangle} = \frac{\langle 32 \rangle}{\langle 3p_1^\flat \rangle}. \quad (5.7)$$

Expressions containing the intermediate shifted momentum $p'_{2,3}$ can be simplified similar to the massless case [4]

$$|p'_{2,3}{}^\flat-\rangle = |p_{2,3}^\flat-\rangle = \frac{\not{p}_{2,3}|p_1^\flat+\rangle}{\langle p_{2,3}^\flat p_1^\flat \rangle} = \frac{\not{p}_{2,3}\not{p}_1|2-\rangle}{\langle p_{2,3}^\flat-|\not{p}_1|2-\rangle}, \quad |p'_{2,3}{}^\flat+\rangle = |p_{2,3}^\flat+\rangle = \frac{\not{p}_{2,3}|2-\rangle}{[p_{2,3}^\flat 2]}. \quad (5.8)$$

A particular compact form of the scalar amplitudes has been obtained in [13]:

$$A_n(\phi_1^+, g_2^+, \ldots, \bar{\phi}_n^-) = 2^{n/2-1} im^2 \frac{\langle 2+|\prod_{j=3}^{n-2} (y_{1,j} - \not{p}_j \not{p}_{1,j-1})|(n-1)-\rangle}{y_{1,2} y_{1,3} \ldots y_{1,n-2} \langle 23 \rangle \langle 34 \rangle \ldots \langle (n-2)(n-1) \rangle}, \quad (5.9)$$

where

$$p_{1,j} = \sum_1^j p_j, \quad y_{1,j} = p_{1,j}^2 - m^2. \quad (5.10)$$

It is an instructive exercise to verify that eq. (5.9) is a solution of eq. (5.4):

$$A_n(\phi_1^+, g_2^+, \ldots, \bar{\phi}_n^-) = A_{n-1}(\phi_1'^+, g_{23}'^+, g_4^+, \ldots, \bar{\phi}_n^-)\frac{i}{p_{2,3}{}^2}A_3(g_{(-23)}'^-, g_2'^+, g_3^+) \tag{5.11}$$

$$= 2^{n/2-1}i^3m^2\frac{\langle p_{2,3}^\flat + |\prod_{j=4}^{n-2}(y_{1,j} - \not{p}_j\not{p}_{1,j-1})|(n-1)-\rangle}{y_{1,3}\ldots y_{1,n-2}\langle p_{2,3}^\flat 4\rangle\langle 45\rangle\ldots\langle(n-2)(n-1)\rangle}\frac{1}{\langle 23\rangle[32]}\frac{[32]^3}{[-p_{2,3}^\flat 3][2(-p_{2,3}^\flat)]}$$

$$= 2^{n/2-1}im^2\frac{\langle 2 + |\not{p}_1\not{p}_{2,3}\prod_{j=4}^{n-2}(y_{1,j} - \not{p}_j\not{p}_{1,j-1})|(n-1)-\rangle[32]^2}{y_{1,3}\ldots y_{1,n-2}\langle 23\rangle\langle 2 + |\not{p}_{2,3}|4+\rangle\langle 45\rangle\ldots\langle(n-2)(n-1)\rangle\langle 2 + |\not{p}_1\not{p}_{2,3}|3-\rangle}$$

$$= 2^{n/2-1}im^2\frac{\langle 2 + |\prod_{j=3}^{n-2}(y_{1,j} - \not{p}_j\not{p}_{1,j-1})|(n-1)-\rangle}{y_{1,2}\ldots y_{1,n-2}\langle 23\rangle\ldots\langle(n-2)(n-1)\rangle}.$$

In the last step we have used the identity [13]

$$\langle 2 + |\not{p}_1\not{p}_{2,3} = \langle 2 + |(y_{1,3} - \not{p}_3\not{p}_{1,2}) \tag{5.12}$$

to extend the product in the numerator down to $j = 3$. This example shows that the shift of a massive quark leads to a computation similar to one for massless particles.

5.2 Amplitudes with one negative helicity gluon adjacent to a massive quark

In this section we consider amplitudes

$$A_n(Q_1^{\lambda_1}, g_2^-, g_3^+, \ldots, g_{n-1}^+, \bar{Q}_n^{\lambda_n}) \tag{5.13}$$

with a pair of massive quarks, a gluon with negative helicity adjacent to a quark and an arbitrary number of positive helicity gluons. As reference spinors for the massive quarks we choose

$$|q_1+\rangle = |q_n+\rangle = |2+\rangle, \quad \langle q_1 + | = \langle q_n + | = \langle 2 + |. \tag{5.14}$$

The light-like momenta $p_1^\flat$ and $p_n^\flat$ associated to p_1 and p_n are given by

$$p_1^\flat = p_1 - \frac{m^2}{2p_1p_2}p_2, \quad p_n^\flat = p_n - \frac{m^2}{2p_2p_n}p_2. \tag{5.15}$$

For the recursion relation we consider the holomorphic shift (3.16) with $(i, j) = (1, 2)$. The spinors are shifted as

$$u_1(-) \to u_1(-) - z|2+\rangle, \quad \langle 2 + | \to \langle 2 + | + z\langle p_1^\flat + |. \tag{5.16}$$

The recursion relation reads

$$A_n(Q_1^{\lambda_1}, g_2^-, g_3^+, \ldots, g_{n-1}^+, \bar{Q}_n^{\lambda_n}) \tag{5.17}$$

$$= \sum_{j=3}^{n-1} A_{n-j+2}(Q_1'^{\lambda_1}, g_{2,j}'^+, g_{j+1}^+, \ldots, g_{n-1}^+, \bar{Q}_n^{\lambda_n})\frac{i}{p_{2,j}^2}A_j(g_{-(2,j)}'^-, g_2'^-, g_3^+, \ldots, g_j^+)$$

where in the j'th term

$$z_j = -\frac{p_{2,j}^2}{\langle p_1^\flat + |\not{p}_{2,j}|2+\rangle}. \tag{5.18}$$

The only ingredients entering the recursion relation (5.17) are the gluonic MHV amplitudes and the quark amplitudes with positive helicity gluons (5.9). The unknown functions do not enter themselves on the right hand side, in contrast to the relations obtained from shifts of gluon momenta [11]. In writing (5.17) we have used that the degree zero gluon amplitudes with more than three external legs vanish on-shell and that the three point degree zero vertex vanishes if an anti-holomorphic spinor is shifted.

From super-symmetric Ward identities we obtain [42]

$$A(Q_1^+, g_2^+, \ldots, g_j^-, \ldots, \bar{Q}_n^+) = 0, \tag{5.19}$$

$$A(Q_1^+, g_2^+, \ldots, g_j^-, \ldots, \bar{Q}_n^-) = \frac{\langle p_n^\flat j \rangle}{\langle p_1^\flat j \rangle} A_n(\phi_1^+, g_2^+, \ldots, g_j^-, \ldots, \bar{\phi}_n^-), \tag{5.20}$$

$$A(Q_1^-, g_2^+, \ldots, g_j^-, \ldots, \bar{Q}_n^+) = -\frac{\langle p_1^\flat j \rangle}{\langle p_n^\flat j \rangle} A_n(\phi_1^-, g_2^+, \ldots, g_j^-, \ldots, \bar{\phi}_n^+). \tag{5.21}$$

Therefore the amplitude for the quark helicities $(Q_1^+, \bar{Q}_n^+)$ vanishes. This follows also from the recursion relation (5.17). In this case the right-hand-side of eq. (5.17) equals zero, since the quark-gluon amplitude with only positive helicity labels vanishes.

Furthermore, eq. (5.20) and eq. (5.21) can be used to relate the helicity combinations $(Q_1^+, \bar{Q}_n^-)$ and $(Q_1^-, \bar{Q}_n^+)$. It follows that only the helicity combinations $(Q_1^\pm, \bar{Q}_n^-)$ need to be considered.

Inserting the explicit expressions for the sub-amplitudes into (5.17) we obtain for the helicity conserving amplitude

$$A_n(Q_1^+, g_2^-, g_3^+, \ldots, g_{n-1}^+, \bar{Q}_n^-) = 2^{n/2-1} i \frac{\langle p_n^\flat 2 \rangle}{\langle p_1^\flat 2 \rangle} \frac{1}{\langle 23 \rangle \ldots \langle (n-2)(n-1) \rangle} \times$$

$$\sum_{j=3}^{n-1} \frac{\langle 2 - |p_1 p_{2,j}|2+ \rangle^2}{p_{2,j}^2 \langle 2 - |p_1 p_{2,j}|j+ \rangle} \left(\delta_{j,n-1} + \delta_{j \neq n-1} \frac{m^2 \langle 2 - |p_{2,j}|\Phi_{j+1,n}- \rangle \langle j(j+1) \rangle}{y_{1,j} \langle 2 - |p_1 p_{2,j}|(j+1)+ \rangle} \right) \tag{5.22}$$

where $\delta_{j \neq n-1} = 1 - \delta_{j,n-1}$ and we used a short-hand notation for the frequently occurring quantity

$$|\Phi_{k,n}- \rangle = \prod_{j=k}^{n-2} \left(1 - \frac{p_j p_{1,j}}{y_{1,j}} \right) |(n-1)- \rangle. \tag{5.23}$$

Intermediate expressions containing spinors of the shifted momentum $p'_{2,j}$ have been simplified according to

$$|p'_{2,j}{}^\flat + \rangle = |p_{2,j}^\flat + \rangle = \frac{p_{2,j} p_1 |2+ \rangle}{\langle p_{2,j}^\flat + |p_1|2+ \rangle}, \quad |p'_{2,j}{}^\flat - \rangle = |p_{2,j}^\flat - \rangle = \frac{p_{2,j} |2+ \rangle}{\langle p_{2,j}^\flat 2 \rangle}. \tag{5.24}$$

Multiplying the result (5.22) by a factor $\langle p_1^\flat 2 \rangle / \langle p_n^\flat 2 \rangle$ results in a new representation of the corresponding amplitude with a pair of massive scalars. Compared to a previous computation of this amplitude in eq. (39) of [11], our result has a similar structure but is simpler since we used the more compact expression (5.9) as input. Furthermore we obtained the result directly from known quantities whereas in a calculation using only shifts of gluons [11] a much more complicated procedure of iterated shifts is necessary.

The helicity flip amplitude is obtained with only small modifications:

$$A_n(Q_1^-, g_2^-, g_3^+, \ldots, g_{n-1}^+, \bar{Q}_n^-) = 2^{n/2-1} i \frac{\langle p_1^\flat p_n^\flat \rangle}{m} \frac{1}{\langle 23 \rangle \ldots \langle (n-2)(n-1) \rangle} \times \qquad (5.25)$$

$$\sum_{j=3}^{n-1} \frac{\langle 2 - |\not p_1 \not p_{2,j}| 2+ \rangle^2}{p_{2,j}^2 \langle 2 - |\not p_1 \not p_{2,j}| j+ \rangle} \left(1 + p_{2,j}^2 \frac{\langle 2 p_n^\flat \rangle}{\langle 2 - |\not p_{2,j} \not p_1^\flat | p_n^\flat + \rangle} \right)$$

$$\times \left(\delta_{j,n-1} + \delta_{j \neq n-1} \frac{m^2 \langle 2 - |\not p_{2,j}| \Phi_{j+1,n} - \rangle \langle j(j+1) \rangle}{y_{1,j} \langle 2 - |\not p_1 \not p_{2,j}| (j+1)+ \rangle} \right).$$

6. Summary and conclusions

In this paper we considered on-shell recursion relations for Born QCD amplitudes. We put particular emphasis on amplitudes with several pairs of quarks and massive quarks and gave a detailed description on how to shift the external particles in spinor space. For massive quarks this implies a particular choice of reference spinors, which define the spin quantisation axis. We found that all Born QCD amplitudes, which have at least some external particles which are not massive quarks, can be computed by on-shell recursion relations using two-particle shifts. Amplitudes with only massive quarks can be computed recursively from three-particle shift. We gave a detailed proof of the validity of the recursion relation. As an application we considered helicity amplitudes including a pair of massive quarks, zero or one negative helicity gluons and an arbitrary number of positive helicity gluons.

Acknowledgments

A. Spinors

We define the light-cone coordinates as

$$p_+ = p_0 + p_3, \quad p_- = p_0 - p_3, \quad p_\perp = p_1 + i p_2, \quad p_{\perp*} = p_1 - i p_2. \qquad (A.1)$$

In terms of the light-cone components of a light-like four-vector, the corresponding massless spinors $\langle p \pm |$ and $|p\pm\rangle$ can be chosen as

$$|p+\rangle = \frac{e^{-i\frac{\phi}{2}}}{\sqrt{|p_+|}} \begin{pmatrix} -p_{\perp*} \\ p_+ \end{pmatrix}, \quad |p-\rangle = \frac{e^{-i\frac{\phi}{2}}}{\sqrt{|p_+|}} \begin{pmatrix} p_+ \\ p_\perp \end{pmatrix},$$

$$\langle p+| = \frac{e^{-i\frac{\phi}{2}}}{\sqrt{|p_+|}} \left(-p_\perp, p_+ \right), \quad \langle p-| = \frac{e^{-i\frac{\phi}{2}}}{\sqrt{|p_+|}} \left(p_+, p_{\perp*} \right), \qquad (A.2)$$

where the phase ϕ is given by

$$p_+ = |p_+| e^{i\phi}. \qquad (A.3)$$

If p_+ is real and $p_+ > 0$ we have the following relations between a spinor corresponding to a vector p and a spinor corresponding to a vector $(-p)$:

$$
\begin{aligned}
|(-p)\,\pm\rangle &= i\,|p\pm\rangle, \\
\langle(-p)\,\pm| &= i\,\langle p\pm|.
\end{aligned}
\tag{A.4}
$$

Therefore the spinors of massive quarks and anti-quarks are related by $u(-k,\pm) = iv(k,\pm)$ and $\bar{u}(-k,\pm) = i\bar{v}(k,\pm)$. The polarisation vectors of the gluons are unchanged under the reversal of the momentum. Spinor products are denoted as

$$
\langle pq\rangle = \langle p-|q+\rangle = p^A q_A, \quad [qp] = \langle q+|p-\rangle = q_{\dot{A}} p^{\dot{A}}.
\tag{A.5}
$$

B. Exceptional cases

For the exceptional cases we consider as an example the helicity configuration (i^+, j^+) with the holomorphic shift. Similar considerations apply to the anti-holomorphic shift and to the configuration (i^-, j^-) with both types of shifts. The exceptional cases have two origins: First, for the helicity configuration (i^+, j^+) with the holomorphic shift we have to consider the cases where we cannot choose to additional particles k and l, such that in the set

$$
\{i, k, l\}
\tag{B.1}
$$

no fermion line connects two of the three external particles. These are the cases listed in eq. (4.8). Secondly, we have to consider the cases, where particle i or particle j is a massive quark or anti-quark and one cannot choose two additional particles k and l with equal helicities. These are the cases listed in eq. (4.22).

The exceptional cases are all limited to amplitudes with no more than six external particles, We discuss these amplitudes case by case. We start with the cases related to eq. (4.8) and discuss at the end the cases of eq. (4.22).

a) The case $A_4(Q, g_i^+, g_j^+, \bar{Q})$: We consider the relevant helicity amplitudes for massive quarks. We have for the unshifted amplitudes

$$
\begin{aligned}
A_4(Q_1^+, g_2^+, g_3^+, \bar{Q}_4^+) &= -2i\frac{m\langle q_1 q_4\rangle}{\langle q_1 p_1^\flat\rangle\langle p_4^\flat q_4\rangle}\frac{[23]}{\langle 23\rangle}\frac{m^2}{2p_1 p_2}, \\
A_4(Q_1^+, g_2^+, g_3^+, \bar{Q}_4^-) &= 2i\frac{\langle q_1 - |p_4|q_4-\rangle}{\langle q_1 p_1^\flat\rangle[p_4^\flat q_4]}\frac{[23]}{\langle 23\rangle}\frac{m^2}{2p_1 p_2}, \\
A_4(Q_1^-, g_2^+, g_3^+, \bar{Q}_4^+) &= -2i\frac{\langle q_1 + |p_1|q_4+\rangle}{[q_1 p_1^\flat]\langle p_4^\flat q_4\rangle}\frac{[23]}{\langle 23\rangle}\frac{m^2}{2p_1 p_2}, \\
A_4(Q_1^-, g_2^+, g_3^+, \bar{Q}_4^-) &= 2i\frac{\langle q_1 + |p_1 p_4|q_4-\rangle}{[q_1 p_1^\flat][p_4^\flat q_4]}\frac{[23]}{\langle 23\rangle}\frac{m}{2p_1 p_2}.
\end{aligned}
\tag{B.2}
$$

The massless case is included as the special case $m = 0$, in which all four helicity amplitudes vanish. For the holomorphic shift we have the substitution

$$
\begin{aligned}
|2+\rangle &\to |2+\rangle - z|3+\rangle, \\
\langle 3+| &\to \langle 3+| + z\langle 2+|.
\end{aligned}
\tag{B.3}
$$

One observes that all non-vanishing helicity amplitudes fall off as $1/z$ for large z due to the factor $1/(2p_1p_2)$. This case does not lead to any restrictions.

b) The case $A_4(Q_i^+, g_j^+, g, \bar{Q})$: This case is already excluded, as i and j are adjacent.

c) The case $A_4(Q_i^+, g, g_j^+, \bar{Q})$: Apart from the helicity amplitudes $A_4(Q_1^+, g_2^+, g_3^+, \bar{Q}_4^+)$ and $A_4(Q_1^+, g_2^+, g_3^+, \bar{Q}_4^-)$, which were already given in eq. (B.2), we need the following two amplitudes:

$$
\begin{aligned}
A_4(Q_1^+, g_2^-, g_3^+, \bar{Q}_4^+) &= 2i \frac{m\langle q_1 q_4 \rangle}{\langle q_1 p_1^\flat \rangle \langle p_4^\flat q_4 \rangle} \frac{\langle 2 - |\not{p}_4|3- \rangle}{2p_1p_2} \left(\frac{\langle q_1 2 \rangle \langle 2 q_4 \rangle}{\langle 23 \rangle \langle q_1 q_4 \rangle} - \frac{\langle 2 - |\not{p}_4|3- \rangle}{s_{23}} \right), \\
A_4(Q_1^+, g_2^-, g_3^+, \bar{Q}_4^-) &= \frac{2i}{\langle q_1 p_1^\flat \rangle [p_4^\flat q_4]} \frac{\langle 2 - |\not{p}_4|3- \rangle}{2p_1p_2} \\
&\quad \times \left(\frac{\langle 2 - |\not{p}_4|3- \rangle \langle q_1 - |\not{p}_4|q_4- \rangle}{s_{23}} - \frac{\langle q_1 2 \rangle \langle 2 - |\not{p}_4|q_4- \rangle}{\langle 23 \rangle} \right). \quad \text{(B.4)}
\end{aligned}
$$

For the holomorphic shift we have $|q_1+\rangle = |3+\rangle$ and the substitution

$$
\begin{aligned}
|p_1^\flat +\rangle &\rightarrow |p_1^\flat +\rangle - z|3+\rangle, \\
\langle 3 + | &\rightarrow \langle 3 + | + z\langle p_1^\flat + |. \quad \text{(B.5)}
\end{aligned}
$$

We observe that all helicity amplitudes go to a constant for $z \rightarrow \infty$. Therefore these helicity amplitudes cannot be computed with the holomorphic shift. As the proof for the recursion relation for the helicity combination (i^+, j^+) is based on induction, we have to exclude for the holomorphic shift all combinations, where particle i is a quark or an anti-quark and particle j is a gluon.

d) Five-parton amplitudes: The five-parton amplitudes of eq. (4.8) are the following:

- The cases $A_5(\bar{Q}_i^+, Q, g_j^+, \bar{Q}', Q')$ and $A_5(\bar{Q}, Q_i^+, g_j^+, \bar{Q}', Q')$;

- The cases $A_{5,sl}(\bar{Q}_i^+, Q, g_j^+; \bar{Q}', Q')$ and $A_{5,sl}(\bar{Q}, Q_i^+, g_j^+; \bar{Q}', Q')$: These are partial amplitudes, where the particles $(\bar{Q}, Q, g)$ form one colour cluster, while the particles $(\bar{Q}', Q)$ form a second colour cluster.

- The cases $A_{5,sl}(\bar{Q}_i^+, Q; g_j^+, \bar{Q}', Q')$ and $A_{5,sl}(\bar{Q}, Q_i^+; g_j^+, \bar{Q}', Q')$: These are partial amplitudes, where the particles $(\bar{Q}', Q', g)$ form one colour cluster and the particles $(\bar{Q}, Q)$ form a second colour cluster.

In view of the conclusions from case c) above, all these cases are already excluded, as particle i is either a quark or an anti-quark, while particle j is a gluon.

e) The cases $A_4(\bar{Q}, Q_i^+, \bar{Q}_j'^+, Q')$ and $A_4(\bar{Q}, Q_i^+, \bar{Q}', Q'_j{}^+)$: We first consider

$A_4(\bar{Q}, Q_i^+, \bar{Q}_j'^+, Q')$. The relevant unshifted amplitudes are:

$$A_4(\bar{Q}_1^-, Q_2^+, \bar{Q}_3'^+, Q_4'^-) = \frac{2i}{\langle q_2 p_2^\flat \rangle [p_1^\flat q_1][q_4 p_4^\flat]\langle p_3^\flat q_3\rangle (p_1+p_2)^2} \tag{B.6}$$

$$\times (\langle q_2 - |\not{p}_2\not{p}_3|q_3+\rangle\langle q_4 + |\not{p}_4\not{p}_1|q_1-\rangle$$
$$-m_2^2\langle q_4 + |\not{p}_4|q_2+\rangle\langle q_1 + |\not{p}_3|q_3+\rangle$$
$$-m_3^2\langle q_2 - |\not{p}_2|q_4-\rangle\langle q_3 - |\not{p}_1|q_1-\rangle + m_2^2 m_3^2\langle q_2 q_3\rangle[q_4 q_1]),$$

$$A_4(\bar{Q}_1^-, Q_2^+, \bar{Q}_3'^+, Q_4'^+) = \frac{2im_3}{\langle q_2 p_2^\flat \rangle [p_1^\flat q_1]\langle q_4 p_4^\flat\rangle\langle p_3^\flat q_3\rangle (p_1+p_2)^2}$$

$$\times (\langle q_2 - |\not{p}_2\not{p}_3|q_3+\rangle\langle q_4 - |\not{p}_1|q_1-\rangle$$
$$+\langle q_2 - |\not{p}_2\not{p}_4|q_4+\rangle\langle q_3 - |\not{p}_1|q_1-\rangle$$
$$+m_2^2\langle q_2 q_4\rangle\langle q_1 + |\not{p}_3|q_3+\rangle + m_2^2\langle q_2 q_3\rangle\langle q_4 - |\not{p}_4|q_1-\rangle),$$

$$A_4(\bar{Q}_1^+, Q_2^+, \bar{Q}_3'^+, Q_4'^-) = \frac{2im_2}{\langle q_2 p_2^\flat \rangle \langle p_1^\flat q_1\rangle[q_4 p_4^\flat]\langle p_3^\flat q_3\rangle (p_1+p_2)^2}$$

$$\times (\langle q_2 - |\not{p}_4|q_4-\rangle\langle q_3 - |\not{p}_3\not{p}_1|q_1+\rangle$$
$$-\langle q_2 - |\not{p}_2\not{p}_3|q_3+\rangle\langle q_4 + |\not{p}_4|q_1+\rangle$$
$$-m_3^2\langle q_2 q_3\rangle\langle q_4 + |\not{p}_1|q_1+\rangle + m_3^2\langle q_3 q_1\rangle\langle q_2 - |\not{p}_2|q_4-\rangle),$$

$$A_4(\bar{Q}_1^+, Q_2^+, \bar{Q}_3'^+, Q_4'^+) = \frac{2im_2 m_3}{\langle q_2 p_2^\flat \rangle \langle p_1^\flat q_1\rangle\langle q_4 p_4^\flat\rangle\langle p_3^\flat q_3\rangle (p_1+p_2)^2}$$

$$\times (-\langle q_2 q_4\rangle\langle q_3 - |\not{p}_3\not{p}_1|q_1+\rangle - \langle q_4 q_1\rangle\langle q_2 - |\not{p}_2\not{p}_3|q_3+\rangle$$
$$-\langle q_2 q_3\rangle\langle q_4 - |\not{p}_4\not{p}_1|q_1+\rangle - \langle q_3 q_1\rangle\langle q_2 - |\not{p}_2\not{p}_4|q_4+\rangle),$$

For the holomorphic shift we have $|q_2+\rangle = |l_3+\rangle$ and $\langle q_3+| = \langle l_2+|$. As a consequence

$$\langle p_2^\flat + | = \langle l_2 + | \text{ and } |p_3^\flat+\rangle = |l_3+\rangle. \tag{B.7}$$

We shift

$$|p_2^\flat+\rangle \rightarrow |p_2^\flat+\rangle - z|l_3+\rangle,$$
$$\langle p_3^\flat + | \rightarrow \langle p_3^\flat + | + z\langle l_2 + |. \tag{B.8}$$

We can summarise the conditions under which the individual helicity amplitudes vanish for $z \rightarrow \infty$ as follows:

$A_4(\bar{Q}_1^-, Q_2^+, \bar{Q}_3'^+, Q_4'^-)$: $m_2 = 0$, $\qquad$ or $\langle q_1 + | = \langle l_2 + |$.

$A_4(\bar{Q}_1^-, Q_2^+, \bar{Q}_3'^+, Q_4'^+)$: $m_2 = 0$, or $m_3 = 0$, or $\langle q_1 + | = \langle l_2 + |$, or $|q_4+\rangle = |l_3+\rangle$.

$A_4(\bar{Q}_1^+, Q_2^+, \bar{Q}_3'^+, Q_4'^-)$: $m_2 = 0$.

$A_4(\bar{Q}_1^+, Q_2^+, \bar{Q}_3'^+, Q_4'^+)$: $m_2 = 0$, or $m_3 = 0$, $\qquad$ or $|q_4+\rangle = |l_3+\rangle$.

We are interested in computing for the combination (i^+, j^+) all helicity combinations with respect to the remaining particles. Therefore the common requirement is $m_2 = 0$. In other words, for the combination $(q_i^+, \bar{q}_j^+)$ the case where particle i is a massive quark has to be excluded.

If we now consider the case $A_4(\bar{Q}, Q_i^+, \bar{Q}', Q'^{+}_j)$, we find in complete analogy again the requirement $m_2 = 0$. Therefore we also exclude the combination (q_i^+, q_j^+) where particle i is a massive quark.

f) The case $A_4(Q_j^+, g, g_i^+, \bar{Q})$: This is an additional case related to eq. (4.22). We are only interested in the case, where the two additional particles have opposite helicities. These are the amplitudes $A_4(Q_1^+, g_2^+, g_3^+, \bar{Q}_4^-)$ and $A_4(Q_1^+, g_2^-, g_3^+, \bar{Q}_4^+)$. One easily shows that both amplitudes vanish as $1/z^2$ for $z \to \infty$.

g) The case $A_4(Q_j^+, g_i^+, g, \bar{Q})$: This is again a case related to eq. (4.22). We are only interested in the case, where the two additional particles have opposite helicities. These are the amplitudes $A_4(Q_1^+, g_2^+, g_3^+, \bar{Q}_4^-)$ and $A_4(Q_1^+, g_2^+, g_3^-, \bar{Q}_4^+)$. Both amplitudes vanish as $1/z$ for $z \to \infty$.

h) The cases $A_5(q_i^+, Q'^{+}_j, q^-, Q'^{\pm}, g^{\mp})$ and $A_6(q_i^+, Q'^{+}_j, q^-, Q'^{\pm}, Q''^{\mp}, Q''^{\mp})$. These cases are again related to eq. (4.22). There are several partial amplitudes which we would have to consider. In this case it is simpler to discuss groups of Feynman diagrams and show that they vanish in the limit $z \to \infty$. We group the Feynman diagrams contributing to $A_5(q_i^+, Q'^{+}_j, q^-, Q'^{\pm}, g^{\mp})$ or $A_6(q_i^+, Q'^{+}_j, q^-, Q'^{\pm}, Q''^{\mp}, Q''^{\mp})$ into three sets: Set 1 consists of all diagrams, where the z-dependence flows through only one propagator. Set 2 consists of all diagrams, where the z-dependence flows through more than one propagator and which do not contain a three-gluon vertex. Finally, set 3 consists of all diagrams which contain a three-gluon vertex.

With arguments similar to the ones given in case e) and f) one shows that the contribution from set 1 vanishes for $z \to \infty$. To see this, note that the five and six-point diagrams in set 1 can be obtained from the four-quark amplitudes discussed previously by setting $m_2 = 0$ and replacing one of the external spinors by an off-shell quark current.

The contribution from set 2 vanishes for $z \to \infty$ since there are at least two z-dependent propagators and no z-dependent vertices. Finally, a short calculation reveals that also the contribution from set 3 vanishes for $z \to \infty$.

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